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Dissipative Collapse in the Presence of Λ

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Abstract We present the general junction conditions for the smooth matching of a spherically symmetric, shear-free spacetime to Vaidya's outgoing metric across a four-dimensional time-like hypersurface in the presence of a cosmological constant. These results generalise earlier treatments by Santos and co-workers on radiating stellar models. We study the thermal evolution of a particular radiating model within the framework of extended irreversible thermodynamics.

Keywords Gravitational collapse · Cosmological constant · Irreversible thermodynamics

1 Introduction

The final outcome of the continued gravitational collapse of a star has continued to attract debate amongst researchers working in this field. With the pioneering effort of Oppenheimer and Snyder [1] in which they considered the gravitational collapse of a dust ball, the issue of late time evolution of a collapsing star has seen a wide range of physically tractable models appearing in the literature. The collapsing dust sphere has been generalised to include charge, the cosmological constant and pressure. With the discovery of the Vaidya solution [2], it is possible to include the effects of dissipation such as a radial heat flux. The junction conditions required for the smooth matching of the interior of the radiating star to Vaidya's outgoing solution was first provided by Santos [3]. These junction conditions were subsequently generalised to include the electromagnetic field as well as shear [4, 5]. The physical properties of these models have been extensively studied by Herrera and co-workers. In particular, relaxational effects on the evolution of the temperature profiles and

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luminosity distributions were shown to be physically viable in these exact models [6]. Recent treatments of dissipative collapse have addressed the role played by the heat flux on the dynamics of the radiating star using a full causal approach [8].

In this paper we seek to model a radiating star when the Einstein field equations are generalised to include the cosmological constant. Observations of the cosmic microwave background, high redshift supernovae and galaxy surveys favour a small value for the cosmological constant. On the other hand, physics predicts much higher values of the cosmological constant. This discrepancy in itself poses challenges to standard cosmological theories and fundamental physics as argued in [9]. The cosmological constant plays an integral role in nonstandard cosmological models of the universe which include the inflationary cold dark matter model (LCDM), some brane world scenarios as well as the idea of a multiverse [10]. The role of the cosmological constant in modelling stars and studying the collapse of bounded configurations has received widespread attention [12–14]. It was shown that the presence of the cosmological constant during gravitational collapse can lead to the formation of black holes and naked singularities. Furthermore, the existence of naked singularities remain stable in the presence of a nonvanishing cosmological constant [13]. To this end we investigate the role played by the cosmological constant during dissipative gravitational collapse. We present the junction conditions at the boundary of the star by matching the generalised Vaidya solution to the interior spacetime. The generalised junction conditions reduce to those presented by Santos [3] when the cosmological constant vanishes.

2 Interior Spacetime

In this section we consider the Einstein field equations for spherically symmetric, shear–free spacetimes in which the cosmological constant is taken to be nonzero. The Einstein field equations governing the interior spacetime with the cosmological constant are

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = T_{ab} \tag{1}$$

where Λ is a constant. The interior spacetime is described by

$$ds^2 = -A^2 dt^2 + B^2 \left[dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right] \tag{2}$$

where $A = A(t,r)$ and $B = B(t,r)$ are metric functions, yet to be determined. For our model the energy-momentum tensor for the interior of the stellar fluid becomes

$$T_{ab} = (\rho + p) u_a u_b + p g_{ab} + q_a u_b + q_b u_a. \tag{3}$$

The fluid four–velocity $\mathbf{u}$ is comoving and is given by

$$u^a = \frac{1}{A} \delta_0^a. \tag{4}$$

The heat flow vector takes the form

$$q^a = (0, q, 0, 0) \tag{5}$$

since $q^a u_a = 0$ and the heat is assumed to flow in the radial direction on physical grounds because of spherical symmetry. The fluid collapse rate $\Theta = u^a_{;a}$ of the stellar model is given by

$$\Theta = 3\frac{\dot{B}}{AB} \tag{6}$$

where dots represent differentiation with respect to t. The Einstein field equations (1) reduce to

$$\rho = 3\frac{1}{A^2}\frac{\dot{B}^2}{B^2} - \frac{1}{B^2}\left(2\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{4}{r}\frac{B'}{B}\right) - \Lambda, \tag{7}$$

$$p = \frac{1}{A^2}\left(-2\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right) + \frac{1}{B^2}\left(\frac{B'^2}{B^2} + 2\frac{A'}{A}\frac{B'}{B} + \frac{2}{r}\frac{A'}{A} + \frac{2}{r}\frac{B'}{B}\right) + \Lambda, \tag{8}$$

$$p = -2\frac{1}{A^2}\frac{\ddot{B}}{B} + 2\frac{\dot{A}}{A^3}\frac{\dot{B}}{B} - \frac{1}{A^2}\frac{\dot{B}^2}{B^2} + \frac{1}{r}\frac{A'}{A}\frac{1}{B^2} + \frac{1}{r}\frac{B'}{B^3} + \frac{A''}{A}\frac{1}{B^2} - \frac{B'^2}{B^4} + \frac{B''}{B^3} + \Lambda, \tag{9}$$

$$q = -\frac{2}{AB^2}\left(-\frac{\dot{B}'}{B} + \frac{B'\dot{B}}{B^2} + \frac{A'}{A}\frac{\dot{B}}{B}\right) \tag{10}$$

where dots and primes denote differentiation with respect to time t and r respectively. Note from (10) that the cosmological constant Λ does not directly appear in the expression for the heat flow q.

3 Exterior Spacetime

The Vaidya solution with cosmological constant Λ is given by

$$ds^2 = -\left(1 - \frac{2m(v)}{\mathsf{r}} - \frac{1}{3}\Lambda\mathsf{r}^2\right)dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{11}$$

which describes the exterior spacetime of the collapsing fluid. The Ricci scalar for the line element (11) is simply given

$$R = 4\Lambda$$

whereas in the pure Vaidya solution the Ricci scalar vanishes. Consequently we may write

$$G_{ab} + \Lambda g_{ab} = -\frac{2}{\mathsf{r}^2}\frac{dm}{dv}\delta_a^0\delta_b^0 \tag{12}$$

for the line element (11) with cosmological constant Λ. The Einstein field equations governing the exterior spacetime with cosmological constant Λ are

$$\begin{aligned} R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} &= T_{ab}, \\ G_{ab} + \Lambda g_{ab} &= \Phi^2 k_a k_b \end{aligned} \tag{13}$$

where $T_{ab}=\Phi^2 k_a k_b$ is the energy–momentum tensor of radiation. With the help of (12) the field equations (13) reduce to

$$\Phi^2 = -\frac{2}{\mathsf{r}^2}\frac{dm}{dv}$$

for the exterior spacetime.

4 A Radiating Model

In order to obtain a complete description of a radiating star, the interior spacetime must be smoothly matched to the exterior spacetime. The junction conditions have been extensively studied since the seminal results of Santos [3] in which he considered the matching of a general spherically symmetric, shear-free line element to the outgoing Vaidya spacetime. To this end we present only the main results for the matching of the line element (2) to the line element (11) which are

$$A(r_\Sigma,t)dt = \left(1-\frac{2m}{\mathsf{r}_\Sigma}-\frac{1}{3}\Lambda \mathsf{r}^2{}_\Sigma + 2\frac{d\mathsf{r}_\Sigma}{dv}\right)^{\frac{1}{2}} dv, \tag{14}$$

$$r_\Sigma B(r_\Sigma,t) = \mathsf{r}_\Sigma(v), \tag{15}$$

$$m(v) = \left(\frac{r^3 B}{2A^2}\dot{B}^2 - r^2 B' - \frac{r^3}{2B}B'^2 - \frac{1}{6}\Lambda r^3 B^3\right)_\Sigma, \tag{16}$$

$$p_\Sigma = (qB)_\Sigma. \tag{17}$$

The junction conditions (14)–(17) generalise the junction conditions of Santos [3] to include the cosmological constant. The condition (17) corresponds to the conservation of momentum flux across the hypersurface Σ. We now turn our attention to seeking an analytical model of a radiating star which satisfy the Einstein field equations and the junction conditions. We analyse a spherically symmetric relativistic radiating star undergoing shear-free gravitational collapse. We further assume that the particle trajectories within the stellar core are geodesics. The acceleration vanishes in this limit. The line element (2) reduces to

$$ds^2 = -dt^2 + B^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{18}$$

where the metric function $B=B(r,t)$ is yet to be determined. The Einstein field equations, for the line element (18) and the energy momentum tensor (3), reduce to

$$\rho = 3\frac{\dot{B}^2}{B^2} - \frac{1}{B^2}\left(2\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{4}{r}\frac{B'}{B}\right) - \Lambda, \tag{19}$$

$$p = -2\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + \frac{1}{B^2}\left(\frac{B'^2}{B^2} + \frac{2}{r}\frac{B'}{B}\right) + \Lambda, \tag{20}$$

$$p = -2\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + \frac{1}{B^2}\left(\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{1}{r}\frac{B'}{B}\right) + \Lambda, \tag{21}$$

$$q = -\frac{2}{B^2}\left(-\frac{\dot{B}'}{B} + \frac{B'\dot{B}}{B^2}\right). \tag{22}$$

Imposing pressure isotropy within the stellar interior requires equating (20) and (21) which yields

$$\left(\frac{1}{B}\right)'' = \frac{1}{r}\left(\frac{1}{B}\right)'. \tag{23}$$

Equation (23) is easily integrable and we obtain

$$B = \frac{d}{C_2(t) - C_1(t)r^2} \tag{24}$$

where $C_1(t)$ and $C_2(t)$ are functions of integration, and d is a constant. The temporal evolution of our model is determined by invoking junction condition (17) which reduces to

$$\begin{aligned} &-4db(\dot{C}_1C_2 - C_1\dot{C}_2)(C_1b^2 - C_2) - 4C_1C_2(C_1b^2 - C_2)^2 \\ &\quad - 2d^2(\ddot{C}_1b^2 - \ddot{C}_2)(C_1b^2 - C_2) + 5d^2(\dot{C}_1b^2 - \dot{C}_2)^2 - \Lambda d^2(C_2b^2 - C_2)^2 = 0 \end{aligned} \tag{25}$$

where $r = b$ determines the boundary of the star. Equation (25) governs the temporal behaviour of our model in which the fluid trajectories are geodesic. Introducing the transformation

$$C_1b^2 - C_2 = u(t) \tag{26}$$

(25) reduces to

$$4bdu^2\dot{C}_1 + 4(u^2 - bd\dot{u})uC_1 - 4b^2u^2C_1^2 = d^2(2u\ddot{u} - 5\dot{u}^2 + \Lambda u^2). \tag{27}$$

As demonstrated by Thirukkannesh and Maharaj [15], (27) is a Riccati equation (in C_1), and is difficult to solve in general. To complete the integration of (27), in terms of elementary functions, we consider the following special case: When $u = \alpha$, (27) reduces to

$$\dot{C}_1 + \frac{\alpha}{bd}C_1 - \frac{b}{d}C_1^2 = \frac{d\Lambda}{4b}. \tag{28}$$

Making use of the transformation

$$C_1 = \frac{-d\dot{w}}{bw} \tag{29}$$

(28) becomes a second order differential equation with constant coefficients,

$$\ddot{w} + \frac{\alpha}{bd}\dot{w} - \frac{\Lambda}{4}w = 0, \tag{30}$$

which has solution

$$w = A_1 \exp\left[\frac{1}{2}\left(\lambda - \frac{\alpha}{bd}\right)t\right] + A_2 \exp\left[-\frac{1}{2}\left(\lambda + \frac{\alpha}{bd}\right)t\right] \tag{31}$$

where $\lambda = \sqrt{\frac{\alpha^2}{b^2d^2} + \Lambda}$ and A_1 and A_2 are constant of integration. Hence

$$C_1 = \left(-\frac{d}{2b}\right)\frac{A_1(\lambda - \frac{\alpha}{bd}) - A_2(\lambda + \frac{\alpha}{bd})\exp[-\lambda t]}{A_1 + A_2\exp[-\lambda t]}. \tag{32}$$

For this case the metric (18) takes the form

$$ds^2 = -dt^2 + \frac{d^2}{[C_1(b^2 - r^2) - \alpha]^2}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{33}$$

where C_1 is given by (32). When the cosmological constant $\Lambda = 0$ the line element (33) reduce to

$$ds^2 = -dt^2 + \frac{b^4 d^2}{\alpha^2}\left[\frac{A_2 + \exp(\frac{\alpha(t+a)}{bd})}{A_2 r^2 + b^2 \exp(\frac{\alpha(t+a)}{bd})}\right]^2 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{34}$$

where we have set $A_1 = \exp[\frac{\alpha a}{bd}]$. The metric (34) is a particular case of Thirukkanesh and Maharaj [15] model.

5 Thermodynamical Behaviour

In this section we investigate the influence of the cosmological constant on the thermal evolution of our radiating stellar model. Utilising the line element (33) when $\alpha = bd =$ constant and $A_1 = 1$, we can write

$$ds^2 = -dt^2 + \left[\frac{2bd(A + e^{\lambda t})}{d(b^2 - r^2)[(1+\lambda)A - (\lambda - 1)e^{\lambda t}] - 2b^2 d(A + e^{\lambda t})}\right]^2 \times \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{35}$$

where $\lambda = \sqrt{1 + \Lambda}$ and $A = A_2$. When $A = 0$ and $\Lambda = 0$ then (35) becomes the Minkowski metric. We employ the Maxwell-Cattaneo heat transport equation to study the behvaviour of the causal and noncausal temperatures. The Maxwell-Cattaneo heat transport equation can be written as

$$\tau h_a{}^b \dot{q}_b + q_a = -\kappa\left(h_a{}^b \nabla_b T + T\dot{u}_a\right) \tag{36}$$

where $h_{ab} = g_{ab} + u_a u_b$ projects into the comoving rest space, T is the local equilibrium temperature, κ (≥ 0) is the thermal conductivity, and τ (≥ 0) is the relaxational time-scale which gives rise to the causal and stable behaviour of the theory. To obtain the noncausal Fourier heat transport equation we set $\tau = 0$ in (36). For the metric (18), equation (36) becomes

$$\tau(qB)^{\cdot} + qB = -\frac{\kappa(T)'}{B}. \tag{37}$$

Assuming power-law generalisations for the thermal conductivity κ, the mean collision time between massive and massless particles τ_c and the relaxation time τ, we can write

$$\kappa = \gamma T^3 \tau_c, \qquad \tau_c = \left(\frac{\alpha}{\gamma}\right) T^{-\sigma}, \qquad \tau = \left(\frac{\beta\gamma}{\alpha}\right)\tau_c, \tag{38}$$

where $\alpha \geq 0$, $\beta \geq 0$ and $\sigma \geq 0$ are constants. Note that the mean collision time decreases with growing temperature as expected except for the special case $\sigma = 0$, when it is constant. With the above assumptions the causal heat transport equation (37) reduces to

$$\beta(qB)^{\cdot} T^{-\sigma} + (qB) = -\alpha \frac{T^{3-\sigma}(T)'}{B}. \tag{39}$$

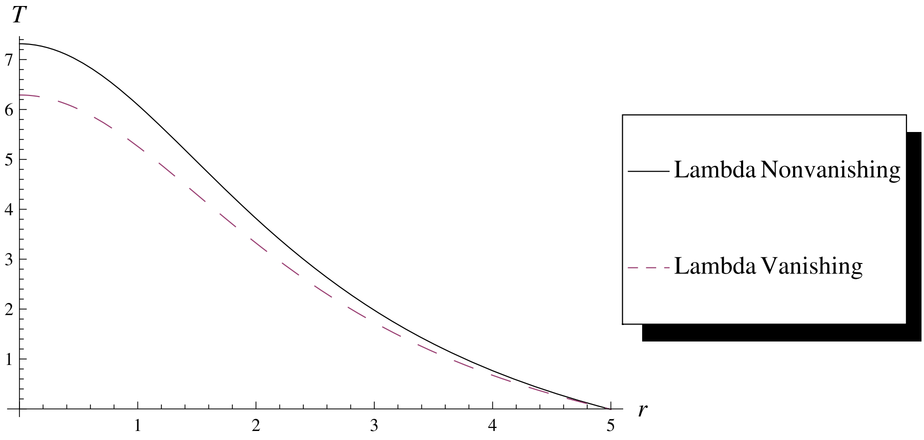

Fig. 1 Noncausal temperature profiles

Solutions to (39) were first obtained by Govinder and Govender [16] in their general treatment. We consider the case of constant mean collision time which corresponds to the case $\sigma = 0$. For our model we are in a position to calculate the causal and noncausal temperatures explicitly. From (39) we obtain the causal temperature profile

$$\begin{aligned} T^4 = {} & \frac{16Ae^{\lambda t}(b^2-r^2)\lambda^2\{Ae^{\lambda t}[2(b^2+r^2)+\beta\lambda^2(b^2-r^2)]\}}{\alpha(A+e^{\lambda t})^2[e^{\lambda t}((b^2-r^2)\lambda+b^2+r^2)+A(b^2+r^2-(b^2-r^2)\lambda)]} \\ & - \frac{16Ae^{\lambda t}(b^2-r^2)\lambda^2\{-e^{2\lambda t}(\beta\lambda-1)[b^2z_1-r^2z_2)]-A^2(1+\beta\lambda)[b^2z_2-r^2z_1]\}}{\alpha(A+e^{\lambda t})^2[e^{\lambda t}((b^2-r^2)\lambda+b^2+r^2)+A(b^2+r^2-(b^2-r^2)\lambda)]} \\ & + T^4_\Sigma \end{aligned} \tag{40}$$

where $z_1 = \lambda + 1$ and $z_2 = \lambda - 1$. The effective surface temperature of a star is given by

$$(\bar{T}^4)_\Sigma = \left(\frac{1}{r^2B^2}\right)\left(\frac{L}{4\pi\delta}\right) \tag{41}$$

where L is the luminosity at infinity and δ (>0) is a constant. The luminosity at infinity can be calculated from

$$L_\infty = -\frac{dm}{dv} \tag{42}$$

where

$$m(v) = \left[\frac{r^3B\dot{B}^2}{2} - r^2B' - \frac{r^3B'^2}{2B}\right]_\Sigma . \tag{43}$$

From (40), we note that the causal and noncausal temperatures coincide at the boundary of the stellar configuration.

Figure 1 illustrates the noncausal temperature profiles for the vanishing and nonvanishing cosmological constant. In this case we assume that the fluid is close to hydrostatic equilibrium and Fig. 1 indicates a reasonable behaviour of the temperature. The plots clearly indicate that an increase in the value of the cosmological constant (≈10% in our investigation)

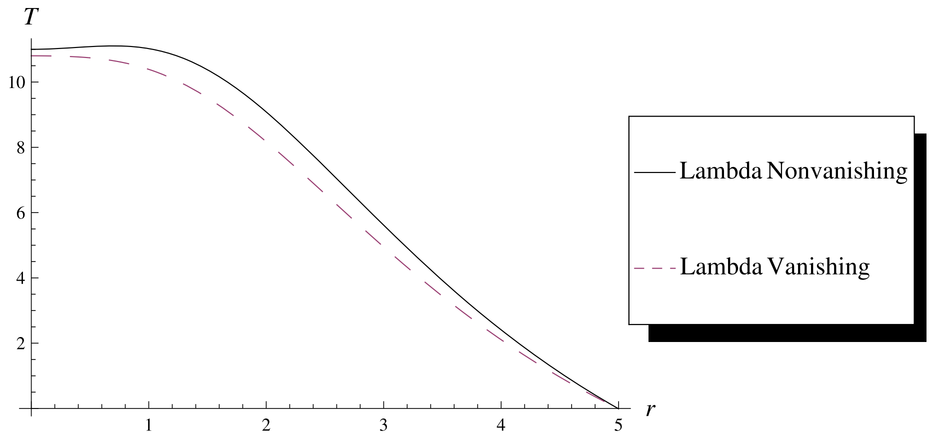

Fig. 2 Causal temperature profiles

leads to an increased temperature throughout the stellar core. Figure 2 shows the contributions from relaxational effects when the star is in quasi-steady hydrostatic equilibrium. This condition corresponds to nonzero values for β in (40). Comparisons with Figs. 1 and 2 indicate that the causal temperature is greater than the noncausal temperature at each interior point within the star. This is in agreement with earlier investigations of relaxational effects on the thermal evolution of a collapsing, radiating star [6, 7].

5.1 Conclusion

We have successfully matched a spherically symmetric, shear-free spacetime to the exterior Vaidya solution with nonzero cosmological constant. We find that the pressure at the boundary of the collapsing star is proportional to the magnitude of the heat flux. This result generalises the junction conditions obtained earlier by Santos. We further investigated the effect of the cosmological constant on the temperature profile of the star by employing the Maxwell-Cattaneo heat transport equation. Our results are in agreement with earlier findings that relaxational effects lead to enhanced temperature gradients within the stellar core [6, 7, 11]. These effects are increased by the presence of the cosmological constant and may become dominant during the latter stages of collapse.

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